16th

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APMO 2004

16th Asia Pacific Mathematical Olympiad
APMO 2004 Problems and Solutions

Problem 1

Find all non-empty finite sets S of positive integers such that if m,n ∈ S, then (m+n)/gcd(m,n) ∈ S.

Solution

Answer: {2}

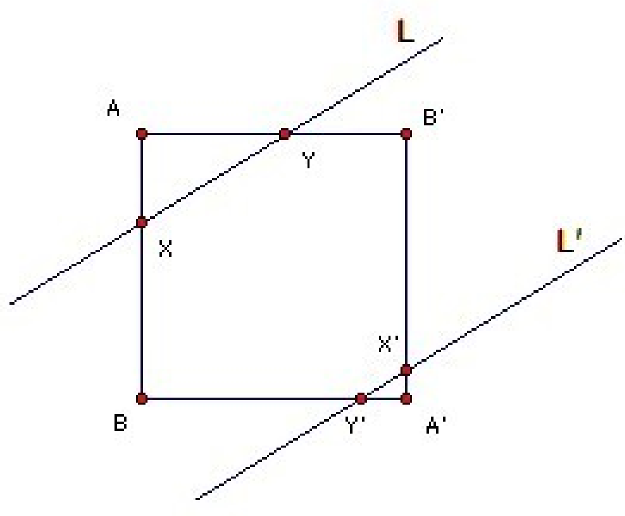
Let k ∈ S. Then (k+k)/gcd(k,k) = 2 ∈ S. Let M be the largest odd element of S. Then (M+2)/gcd(M,2) = M+2 ∈ S. Contradiction. So all elements of S are even.

Let m = 2n be the smallest element of S greater than 2. Then (m+2)/2 = n+1 ∈ S. But n must be > 1 (or m = 2), so 2n > n+1. Hence 2n = 2 (by minimality of m), so n = 1. Contradiction. So S has no elements apart from 2.

Problem 2

A unit square lies across two parallel lines a unit distance apart, so that two triangular areas of the square lie outside the lines. Show that the sum of the perimeters of these two triangles is independent of how the square is placed.

Solution



Let the lines be L, L'. Let the square be ABA'B', with A, A' the two vertices not between L and L'. Let L meet AB at X and AB' at Y. Let L' meet A'B' at X' and A'B at Y'. So AXY and A'X'Y' are similar. Suppose angle AXY = x. If we move L towards A by a distance d perpendicular to itself, then AX is shortened by d cosec x. If L' remains a distance 1 from L, then A'X' is lengthened by d cosec x. The new triangle AXY is similar to the old. Suppose that perimeter AXY = k·AX, then perimeter AXY is increased by kd cosec x. Since AXY and A'X'Y' are similar, perimeter A'X'Y' is shortened by kd cosec x, so the sum of their perimeters is

unchanged. It remains to show that the sum of the perimeters does not depend on the angle x.
Let us move L towards A until L' passes through A', at which point the perimeter of A'X'Y' is zero. Now if h is the height of AXY (from the base XY), then $1 + h = AA' \sin(45^\circ + x) = \sin x + \cos x$. The perimeter of AXY is $h/\sin x + h/\cos x + h/(\sin x \cos x) = h(\sin x + \cos x + 1)/(\sin x \cos x) = (\sin x + \cos x - 1)(\sin x + \cos x + 1)/(\sin x \cos x) = 2$, which is independent of x.

Problem 3

2004 points are in the plane, no three collinear. S is the set of lines through any two of the points. Show that the points can be colored with two colors so that any two of the points have the same color iff there are an odd number of lines in S which separate them (a line separates them if they are on opposite sides of it).

Solution

Let us denote by d_{XY} the number of points separating the points X and Y. If the result is true, then the coloring is effectively determined: take a point X and color it blue. Then for every other point Y, color it blue iff d_{XY} is odd. This will work provided that given any three points A, B, C, we have $d_{AB} + d_{BC} + d_{CA}$ is odd. (For then if Y and Z are the same color, d_{XY} and d_{XZ} have th same parity, so d_{YZ} is odd, which is correct. Similarly, if Y and Z are opposite colors, then d_{YZ} is even, which is correct.)
We are interested in lines which pass through an interior point of one or more of AB, BC, CA. Lines cannot pass through all three. If they pass through two, then they do not affect the parity of $d_{AB} + d_{BC} + d_{CA}$. So we are interested in lines which pass through A and the (interior of) BC, and similarly for B, C. Let $n_1, n_2, \dots , n_7$ be the number of points (excluding A, B, C) in the various regions (as shown). The number of lines through A and BC is $n_1 + n_2 + n_3$. So $d_{AB} + d_{BC} + d_{CA} = (n_1 + n_2 + n_3) + (n_1 + n_4 + n_5) + (n_1 + n_6 + n_7) = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = (2004 - 3) = 1 \mod 2$.
Thanks to Dinu Razvan

Problem 4

Show that $[(n-1)!/(n^2+n)]$ is even for any positive integer n.

Solution

Thanks to Juan Ignacio Restrepo
For n = 1, 2, 3, 4, 5 we have $[(n-1)!/(n^2+n)] = 0$, which is even. So assume $n \ge 6$.
If n and n+1 are composite, then they must divide (n-1)!. They are coprime, so their product must divide (n-1)!. Note that just one of n, n+1 is even. For $m \ge 6$, (m-2)! is divisible by more powers of 2 than m, so $(n-1)!/(n^2+n)$ is even. It remains to consider the two cases n+1 = p, a prime, and n = p a prime.
If n+1 = p, then p-1 is composite, so p-1 divides (p-2)!. Let k = (p-2)!/(p-1). By Wilson's theorem we have $(p-2)! = 1 \mod p$, so $k(p-1) = 1 \mod p$

and hence k = -1 mod p. So (k+1)/p is an integer. But k is even, so k+1 is odd and hence (k+1)/p is odd. Now [k/p] = (k+1)/p - 1, so [k/p] = [(n-1)!/(n^2+n)] is even.

If n = p, then k = (p-1)!/(p+1) is an even integer, so k+1 is an odd integer. By Wilson's theorem, k(p+1) = -1 mod p, so (k+1)/p is an integer and hence an odd integer. Hence [(n-1)!/(n^2+n)] = [k/p] = (k+1)/p - 1 is even.

Wilson's theorem states that p is prime iff (p-1)! = -1 mod p. If p is composite, then it has a factor ≤ p-1, which divides (p-1)! and so does not divide (p-1)! + 1. Hence (p-1)! ≠ -1 mod p. Now suppose p is prime. If p = 2, then (p-1)! = 1 = -1 mod 2. So assume p is odd. Note first that if a^2 = 1 mod p and 0 < a < p, then a = 1 or p-1. For p divides (a+1)(a-1) and so it divides either a+1, giving a = p-1, or it divides a-1, giving a = 1. Now for each 0 < a < p, there is a unique a' such that aa' = 1 mod p. We have just shown that a = a' iff a = 1 or p-1. So we can divide 2, 3, ... , p-2 into pairs with the product of each pair being 1 mod p. Hence (p-2)! = 1 mod p, as required.

On the powers of 2, note that 2, 2^2, 2^3, ... , 2^{k-1} < 2^k and their product is $2^{k(k-1)/2}$. If 2^k < n < 2^{k+1}, then n is divisible by at most 2^{k-1}. So for n ≥ 16, for example, (n-8)!/n is even. For n = 8, 9, ... , 15, (n-2)! is divisible by 2·4·6 and n is divisible by at most 8, so (n-2)!/n is even. Finally, we can easily check n = 6, 7.

Problem 5

Show that (x^2 + 2)(y^2 + 2)(z^2 + 2) ≥ 9(xy + yz + zx) for any positive reals x, y, z.

Solution

Expanding lhs we get $x^2y^2z^2$ + 2(x^2y^2 + y^2z^2 + z^2x^2) + 4(x^2 + y^2 + z^2) + 8. By AM/GM we have 3(x^2 + y^2 + z^2) ≥ 3(xy + yz + zx) and (2x^2y^2 + 2) + (2y^2z^2 + 2) + (2z^2x^2 + 2) ≥ 4(xy + yz + zx). That leaves us needing $x^2y^2z^2$ + x^2 + y^2 + z^2 + 2 ≥ 2(xy + yz + zx) (*). That is hard, because it is not clear how to deal with the xyz on the lhs.

By AM/GM we have (a+b)(b+c)(c+a) ≥ 8abc. So if (u+v-w), (u-v+w), (-u+v+w) are all non-negative, we can put 2a = -u+v+w, 2b = u-v+w, 2c = u+v-w and get uvw ≥ (-u+v+w)(u-v+w)(u+v-w). If u, v, w are positive, then at most one of (u+v-w), (u-v+w), (-u+v+w) is negative. In that case uvw ≥ (-u+v+w)(u-v+w)(u+v-w) is trivially true. So it holds for all positive u, v, w. Expanding, we get u^3 + v^3 + w^3 + 3uvw ≥ uv(u+v) + vw(v+w) + wu(w+u), which is a fairly well-known inequality. Applying AM/GM to u+v etc we get, u^3 + v^3 + w^3 + 3uvw ≥ 2$(uv)^{3/2}$ + 2$(vw)^{3/2}$ + 2$(wu)^{3/2}$. Finally, putting u = $x^{2/3}$, v = $y^{2/3}$, w = $z^{2/3}$, we get x^2 + y^2 + z^2 + 3$(xyz)^{2/3}$ ≥ 2(xy + yz + zx) (**).

Thus we have $x^2y^2z^2$ + x^2 + y^2 + z^2 + 2 ≥ ≥ (x^2 + y^2 + z^2) + 3$(xyz)^{2/3}$, by applying AM/GM to $x^2y^2z^2$, 1, 1, and now (**) gives the required (*).

Thanks to Jacob Tsimerman

Ignore for a moment the case n = m. Suppose that n > m and we have already established the result for n-1. Since m-1+2(n-1) < m-1+2n, and

G does not have m pairs of joined points it must have n-1 pairs of unjoined points.
So let B be a set of n-1 pairs of unjoined points. Let A be the remaining m+1 points. Take any two points P and Q in A and any pair P', Q' in B. If P is not joined to P' and Q is not joined to Q', then we have a contradiction, because we could remove the pair P', Q' from B and replace it by the two pairs P, P' and Q, Q', thus getting n pairs of unjoined points. So we may assume that P is joined to P'. Now remove P from A, leaving m points and mark the pair P', Q' in B as used. We now repeat. Take any two points in the reduced A and any unmarked pair in B and conclude that one of the pair in A is joined to one of the pair in B. Remove the point from A and mark the pair in B. Since n-1 >= m we can continue in this way until we obtain m pairs of joined points (one of each pair in A and the other in B). Contradiction. So the result is true for n also.
It remains to consider the case n = m.
To get the n-1 pairs we use the induction on m. G has m-1+2m = 3m-1 points and does not have either m pairs of joined points or m pairs of unjoined points. We know by induction that f(m-1, m) = m-1+2m-1 = 3m-2, so f(m, m-1) = 3m-2 also. Since m-1+2m > 3m-2, G has either m pairs of joined points or m-1 points of unjoined points. It does not have the former (by assumption) so it must have the latter.
We now proceed as before. So B is a set of m-1 pairs of unjoined points and A the set of the remaining m+1 points. We get m-1 pairs of joined points (one of each pair in A and the other in B). But now we have run out of unmarked pairs in B. However, there are still two points in A. If they are not joined, then we would have a contradiction, because with the pairs in B they would give n pairs of unjoined points. So they are joined and hence give us an mth joined pair. Contradiction.
Comment. This all seems complicated, and certainly much harder work than questions 1-3. So maybe I am missing something. Does anyone have a simpler solution?